On ODEs from PEPA Models

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Abstract

This paper is about theoretical aspects of the ODEs derived from PEPA models. We show the existence, uniqueness and boundedness of solutions of the ODEs. We also present the convergence of the solutions for non-synchronised cases as well as their relationship to the steady state distributions of some Markov chains. Based on the theoretical results, an alternative semantics of mapping PEPA models to ODEs is provided.

1 Introduction

Stochastic process algebras have enjoyed considerable success in quantified analysis over the last decade. Formalisms such as PEPA [1], EMPA [2], TIPP [3] have been widely used to model systems for which performance measures need to be derived.

In stochastic process algebras, the quantified durations associated with activities, are random variables and usually satisfy exponential distributions. So, the underlying stochastic models, upon which quantitative evaluation has relied, are usually Markov chains [4]. There is a problem of *state space explosion* encountered in the calculation of steady state or transient probability distributions of Markov chains. This problem limits the size of the system which can be subjected to analysis. Hillston and her collaborators [4–6] have developed a new approach—continuous state-space approximation—to avoid this problem. This approach results in a set of ordinary differential equations (ODEs), leading to the evaluation of transient and, in the limit, steady state measures.

PEPA and its ODEs have demonstrated successful application in the performance analysis of large scale systems (see [7–11]). However, some new problems have arisen and await solutions. For example, the following questions are natural and important. What are the characteristics of those ODEs and can these characteristics ensure the performance measures? What is the relationship between the ODEs and the Markov chain of the same PEPA model?

We investigate these problems and present some results in this paper. We will show that all the solutions of ODEs derived from PEPA models not only exist, but are unique and bounded. Moreover, for non-synchronised PEPA models, the solutions have limits and can be related to the probability distribution of a CTMC. Based on these theoretical results, we propose the *density evolution*—an alternative ODEs mapping semantics, which describes the system evolution from the point of view of density.

The remainder of this report is structured as follows. In Section 2, we give a brief introduction to the ODEs derived from PEPA models. We present the theoretical results on the existence, uniqueness and boundedness of solutions of those ODEs in Section 3. In Section 4, we show that the solutions of the ODEs derived from non-synchronised PEPA models converge to finite limits and these limits are related to steady state distributions of a Markov chain. Based on the theoretical results, an alternative semantic mapping PEPA models to ODEs is presentd in Section 5. Finally, we conclude the paper in Section 6.

2 ODEs from PEPA Models

PEPA (Performance Evaluation Process Algebra), developed by Hillston in the 1990s, extends classical process algebras such as CCS or CSP, by associating a random variable, representing duration, with every

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action. It can be regarded as a high-level model specification language for low-level stochastic models, which make it suitable for extracting performance measures as well as deducing functional properties of the system.

To avoid the problem of *state space explosion* encountered in the calculation of the steady state or transient probability distribution of Markov Chains under PEPA models, Hillston proposed a radically different approach in [4] from the following two perspective:

- Instead of calculating the probability distribution, choosing a more abstract state representation in terms of state variables, quantifying the types of behaviour evident in the model.
- Assuming that these state variables are subject to continuous rather than discrete change.

This approach results in a set of ordinary differential equations (ODEs), leading to the evaluation of transient, and in the limit, steady state measures. We first quote the following *numerical vector form*—a new state representation created in [4].

Definition 1. ***Numerical Vector Form*** *[4] For an arbitrary PEPA model $\mathcal{M}$ with n component types $C_i, i = 1, 2, \cdots, n$, each with N_i distinct derivatives, the numerical vector form of $\mathcal{M}$, $\mathcal{N}(\mathcal{M})$, is a vector with $N = \sum_{i=1}^{n} N_i$ entries. The entry $N(C_{i_j})$ records how many instances of the jth local derivative of component type C_i are exhibited in the current state.*

We give more preliminary definitions used in [4]. Consider a local derivative D of a sequential component. An activity (α, r) is an *exit activity* of D if D enables (α, r), i.e. there is a transition $D \stackrel{(\alpha,r)}{\longrightarrow}$ in the labelled transition system of D. We denote the set of exit activities of D by $Ex(D)$. Conversely, we denote the set of local derivatives for which (α, r) is an exit activity by $Ex(\alpha, r)$. For *entry activity* of D, we similarly define $En(D)$ as well as $En(\alpha, r)$.

We also follow the assumptions in [4] about the form of the PEPA models: there are no action hiding, no passive actions, no synchronisation within repeated components, and all cooperating components have the same local rate for shared activities.

Now we show the ODEs from PEPA models. Let $N(C_{i_j}, t)$ denote the jth entry of the ith subvector at time t, i.e. the number of instances of the jth local derivative of sequential component C_i. Then [4]

$$\begin{aligned} \frac{dN\left(C_{i_j}, t\right)}{dt} = & -\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)} \left\{N\left(C_{k_l}, t\right)\right\} \\ & + \sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in En(\alpha,r)} \left\{N\left(C_{k_l}, t\right)\right\} \end{aligned} \tag{1}$$

The first term of the right side of (1) records the rates of exit activities while the second term records the ones of entry activities.

PEPA is the first stochastic process algebra to have a true concurrency semantics via the mapping to ODEs. The true concurrency semantics avoids the state space explosion problem and opens the door to vast applications especially for the large scale systems.

Despite the successful applications of PEPA and its ODEs (see [7–11] etc.), PEPA still has to face some problems arising from this new area. Once the ODEs, derived from the PEPA model, are established, many important questions such as the following may simultaneously arise: What are the characteristics of those ODEs and can these characteristics ensure the performance measures? How to derive the performance measures from the ODEs? What is the relationship between the ODEs and the CTMC of the same PEPA model? Moreover, is there any better way in which the differential equations can be derived and what kind of differential equations should be derived from PEPA models, to guarantee certain good characteristics and so that the performance measures are more consistent with the ones of the CTMC? We will investigate some of these problems in the following sections.

3 Existence, Uniqueness and Boundedness of Solutions

In this section, we will show the solutions of (1) not only exist but are unique and bounded.

3.1 Features of ODEs

We repeat the (1) as follows:

$$\begin{aligned}\frac{dN\left(C_{i_j},t\right)}{dt} = & -\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r\times \min_{C_{k_l}\in Ex(\alpha,r)}\left\{N\left(C_{k_l},t\right)\right\} \\ & +\sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r\times \min_{C_{k_l}\in En(\alpha,r)}\left\{N\left(C_{k_l},t\right)\right\}\end{aligned}$$

Suppose the initial values $N\left(C_{i_j},0\right)$ are given. The first term of the right side of (1) records the rates of exit activities while the second term records the ones of entry activities. We notice an important fact: the sum of all exit activities rates equal to the sum of all entry activities rates for each type of component at any time, since the system is closed and there is no exchange with outside environment. This leads to the following

Proposition 1. *For all i,*

$$\sum_j \frac{dN\left(C_{i_j},t\right)}{dt} = 0, \quad \forall t, \tag{2}$$

or

$$\sum_j N\left(C_{i_j},t\right) = \sum_j N\left(C_{i_j},0\right) = N(C_i), \quad \forall t. \tag{3}$$

Remark 1. *Proposition 1 means the ODEs satisfy* Conservation Law*, i.e. the number of each kind of component keeps unchanged all the time.*

Another important fact to note is: the "exit rates" of C_{i_j} in (1),

$$-\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r\times \min_{C_{k_l}\in Ex(\alpha,r)}\left\{N\left(C_{k_l},t\right)\right\},$$

is related to the number of C_{i_j}. According to the semantics of mapping PEPA models to ODEs, the exit activities of C_{i_j} depend on either C_{i_j} itself or the synchronisation in which C_{i_j} takes part. That means $C_{i_j}\in Ex(\alpha,r)$ as long as $(\alpha,r)\in Ex(C_{i_j})$. In other words,

$$N(C_{i_j},t)\in\left\{N\left(C_{k_l},t\right)\right\}_{C_{k_l}\in Ex(\alpha,r)}. \tag{4}$$

This implies

$$\min_{C_{k_l}\in Ex(\alpha,r)}\left\{N\left(C_{k_l},t\right)\right\}\le N(C_{i_j},t). \tag{5}$$

So we have the following

Proposition 2. *For all C_{i_j},*

$$\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r\times \min_{C_{k_l}\in Ex(\alpha,r)}\left\{N\left(C_{k_l},t\right)\right\}\le \sum_{(\alpha,r)\in Ex(\alpha,r)} r\times N(C_{i_j},t) \tag{6}$$

Proposition 1 and Proposition 2 are two important features of ODEs derived from PEPA models. These good properties strongly connect the Markov chains under the PEPA models and hence result in good analytic results as well as leading us to further understanding of large scale PEPA models.

3.2 Existence, Uniqueness and Boundedness of Solutions

Theorem 1. *The solutions of (1) exit, and are unique given the initial values.*

Proof. Notice "min" is a Lipschitz function (see the proof in Appendix B), by Theorem A2 in Appendix A, we know Theorem 1 holds. □

The above Theorem 1 illustrates the solutions of ODEs from PEPA models both exist and are unique. The following theorem will further show that the solutions are bounded.

Theorem 2. *If $N\left(C_{i_j}, t\right)$ satisfy (1) with nonnegative initial values, then*

$$0 \leq N\left(C_{i_j}, t\right) \leq N(C_i), \quad \forall t. \tag{7}$$

Moreover, if the initial values are positive, then the solutions are positive all the time, i.e.,

$$0 < N\left(C_{i_j}, t\right) \leq N(C_i), \quad \forall t. \tag{8}$$

Proof. By Proposition 1, $\sum_j N\left(C_{i_j}, t\right) = N(C_i)$ for all t, all that is left to do is to prove that $N(C_{i_j})$ is positive or nonnegative. The proof is divided into two cases.

Case 1: Suppose all the initial values are positive, i.e. $\min_{i_j}\left\{N(C_{i_j}, 0)\right\} > 0$.
We will show, for all $t \geq 0$,

$$\min_{i_j}\left\{N(C_{i_j}, t)\right\} > 0.$$

Otherwise, if there exists a $t > 0$ such that $\min_{i_j}\left\{N(C_{i_j}, t)\right\} \leq 0$, then there definitely exists a point $t > 0$ such that

$$\min_{i_j}\left\{N(C_{i_j}, t)\right\} = 0.$$

Let

$$t^* = \inf_{t>0}\left\{\min_{i_j}\left\{N(C_{i_j}, t)\right\} = 0\right\},$$

then $0 < t^* < \infty$. Without loss of generality, we assume $N(C_{1_1}, t)$ reach 0 at t^*, i.e.,

$$N(C_{1_1}, t^*) = 0, N(C_{i_j}, t^*) \geq 0 (i_j \neq 1_1)$$

and

$$N(C_{i_j}, t) > 0, \quad t \in [0, t^*), \quad \forall i, j.$$

Thus, for $t \in [0, t^*]$, by Proposition 1,

$$\begin{aligned}
\frac{dN(C_{1_1}, t)}{dt} &= -\sum_{(\alpha,r)\in Ex\left(C_{1_1}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)}\left\{N\left(C_{k_l}, t\right)\right\} \\
&\quad + \sum_{(\alpha,r)\in En\left(C_{1_1}\right)} r \times \min_{C_{k_l}\in En(\alpha,r)}\left\{N\left(C_{k_l}, t\right)\right\} \\
&\geq -\sum_{(\alpha,r)\in Ex\left(C_{1_1}\right)} r \times N(C_{1_1}, t) \\
&= -N(C_{1_1}, t)\sum_{(\alpha,r)\in Ex\left(C_{1_1}\right)} r.
\end{aligned}$$

Set $R = \sum_{(\alpha,r)\in Ex\left(C_{1_1}\right)} r$, then

$$\frac{dN(C_{1_1}, t)}{dt} \geq -RN(C_{1_1}, t),$$

or

$$\frac{dN(C_{1_1}, t)}{N(C_{1_1}, t)} \geq -Rdt,$$

which implies

$$N(C_{1_1}, t^*) \geq N(C_{1_1}, 0)e^{-Rt^*} > 0.$$

This is a contradiction to $N(C_{1_1}, t^*) = 0$. Thus,

$$0 < N\left(C_{i_j}, t\right) \leq N(C_i), \quad \forall t.$$

Case 2: Suppose $\min_{i_j}\left\{N(C_{i_j},0)\right\}=0$.
Let $u_\delta(i_j,0)=N(C_{i_j},0)+\delta$ and $u_\delta(i_j,t)$ satisfy (1), by the proof of Case 1, $u_\delta(i_j,t)>0$. Note that $\min(\cdot)$ is a Lipschitz function. Then by the Fundamental Inequality in Appendix A, we have

$$|u_\delta(i_j,t)-N(C_{i_j},t)|\leq \delta e^{Kt}, \tag{9}$$

where K is a Lipschitz constant. Thus, for any given $t\geq 0$,

$$N(C_{i_j},t)\geq u_\delta(i_j,t)-\delta e^{Kt}>-\delta e^{Kt}. \tag{10}$$

Let $\delta\downarrow 0$ in (10), then we have $N(C_{i_j},t)\geq 0$. The proof is completed. □

Remark 2. *This theorem shows that the ODEs derived from PEPA models reflect the evolution of realistic systems. Each type of components just changes its states while keeping the total number unchanged. Moreover, the number of the components staying in a particular state at any time is between 0 and the total number. Thus, this evolution can be considered as a density evolution in some sense (see Section 5 for details).*

Remark 3. *The method of this proof can also be applied to the ODEs in [5, 6], which are derived by different mapping semantics.*

4 Convergence of Solutions: without Synchronisation

Now we consider the PEPA models without synchronisation. For this special class of PEPA models, we will show that the solutions of the derived ODEs have finite limits.

4.1 Features of ODEs

Suppose the PEPA model has no synchronisation involved. Without loss of generality, we suppose there is only one kind of component C in the system. In fact, if there are several types of components in the system, the ODEs related to different types of components can be separated and can be treated independently since there is no any interaction between any different kinds of components. In other words, to deal with other types of components, we only need to repeat the process of dealing with type C. Thus, we assume there is only one kind of component C in the system and C has k states: $C_1,C_2,\cdots,C_k$. Then (1) becomes

$$\begin{aligned}\frac{dN\left(C_j,t\right)}{dt}=&-\sum_{(\alpha,r)\in Ex(C_j)} r\times N\left(C_l,t\right)\\ &+\sum_{(\alpha,r)\in En(C_j)} r\times N\left(C_l,t\right)\end{aligned} \tag{11}$$

where $j=1,2,\cdots,k$. Since (11) are linear ODEs, we may rewrite (11) as the following matrix forms:

$$\frac{d\left(N(C_1,t),\cdots,N(C_k,t)\right)}{dt}=\left(N(C_1,t),\cdots,N(C_k,t)\right)Q, \tag{12}$$

where $Q=(q_{ij})$ is a $k\times k$ matrix. Q has many good properties.

Proposition 3. *$Q=(q_{ij})_{k\times k}$ in (12) is an infinitesimal generator matrix*[1]*, which means $(q_{ij})_{k\times k}$ satisfy*

1. *$0\leq -q_{ii}<\infty$ for all i;*

[1]We point out that this infinitesimal generator matrix $Q_{k\times k}$ may not be the infinitesimal generator matrix of the CTMC (we call it original CTMC for convenience) underlying the PEPA model. In fact, the original CTMC has a state space with k^N states and the dimension of its infinitesimal generator matrix should be $k^N\times k^N$, where N is th total number of components in the system. It is obvious that only when $N=1$ they coincide.

From another point of view, however, this $Q_{k\times k}$ is just the infinitesimal generator matrix of the CTMC derived from one copy of the PEPA component. To make a difference from the original one, we consider this CTMC as a "standard" CTMC.

2. $q_{ij} \geq 0$ *for all* $i \neq j$*;*

3. $\sum_{j=1}^{k} q_{ij} = 0$ *for all* i*.*

Proof. It is easy to see item 1 holds; by Proposition 2, item 2 holds. We only prove item 3. By Proposition 1,

$$\frac{dN(C_1)}{dt} + \frac{dN(C_2)}{dt} + \cdots + \frac{dN(C_k)}{dt} = 0, \quad \forall t,$$

which means

$$N(C_1,t)\sum_{j=1}^{k} q_{1j} + N(C_2,t)\sum_{j=1}^{k} q_{2j} + \cdots + N(C_k,t)\sum_{j=1}^{k} q_{kj} = 0, \quad \forall t. \tag{13}$$

The above (13) implies $\sum_{j=1}^{k} q_{ij} = 0$ for all i. □

Remark 4. *Infinitesimal generator matrix is an essential concept in CTMC theory. It represents the rate transition of CTMC and from it the stationary distribution can be derived.*

Remark 5. *This proposition shows that the ODEs derived from PEPA models are strongly related to some certain probabilistic structures such as infinitesimal generator matrix. If there is only one component in the system, then (12) becomes the probability distribution evolution equations of a CTMC which has the rate transition matrix* Q*. It leads us to use the Markov theory to prove the analytic results.*

4.2 Convergence of the ODEs' Solutions

We know, by Theorem 2, the solutions of (11) are bounded between 0 and the total number of component C. It is natural for us to ask whether these solutions converge to finite limits as time goes to infinity. The following theorem will give us a positive answer.

Theorem 3. *Suppose* $N(C_j,t)\,(j=1,2,\cdots,k)$ *satisfy (11), then for any given initial values* $N(C_j,0) \geq 0 (j=1,2,\cdots,k)$*, we have*

$$\lim_{t\to\infty} N(C_j,t) = N(C_j,\infty), \quad j=1,2,\cdots,k. \tag{14}$$

Proof. Construct a CTMC [2] which has state space $S = \{C_1, C_2, \cdots, C_k\}$, infinitesimal generator matrix Q in (12) and initial probability distribution $\pi(C_j,0) = \frac{N(C_j,0)}{N} (j=1,2,\cdots,k)$. Then according to Markov theory ([12], page 52), $\pi(C_j,t)(j=1,2,\cdots,k)$, the probability distribution of this new CTMC at time t satisfy

$$\frac{d\left(\pi(C_1,t),\cdots,\pi(C_k,t)\right)}{dt} = \left(\pi(C_1,t),\cdots,\pi(C_k,t)\right)Q \tag{15}$$

Since the new CTMC is irreducible, it has the steady state distribution, i.e.,

$$\lim_{t\to\infty} \pi(C_j,t) = \pi(C_j,\infty), \quad j=1,2,\cdots,k. \tag{16}$$

Notice $\frac{N(C_j,t)}{N}$ also satisfy (15) and the initial values $\frac{N(C_j,0)}{N}$ equal to $\pi(C_j,0)$, by the uniqueness of the solutions of (15), we have

$$\frac{N(C_j,t)}{N} = \pi(C_j,t), \quad j=1,2,\cdots,k, \tag{17}$$

and hence,

$$\lim_{t\to\infty} N(C_j,t) = N\pi(C_j,\infty), \quad j=1,2,\cdots,k. \tag{18}$$

□

[2] This CTMC is essentially the same to the "standard" CTMC mentioned in footnote 1.

Remark 6. *If there are m types of components in the system: $C_1, C_2, \cdots, C_m$, and have $k_1, k_2, \cdots, k_m$ component states respectively, since there is no interaction between different types of components the derived ODEs can be written in the following forms:*

$$\frac{d\left(N(C_{i_1},t),\cdots,N(C_{i_{k_i}},t)\right)}{dt} = \left(N(C_{i_1},t),\cdots,N(C_{i_{k_i}},t)\right) Q_{k_i\times k_i}, \tag{19}$$

where $i = 1, 2, \cdots, m$. Moreover, each $Q_{k_i\times k_i}(i = 1, 2, \cdots, m)$ is an infinitesimal generator matrix. By Theorem 3, for type C_i we have

$$\lim_{t\to\infty} N(C_{i_j},t) = N\pi(C_{i_j},\infty), \quad j = 1, 2, \cdots, k_i, \tag{20}$$

where $\{\pi(C_{i_j}\}_{j=1,2,\cdots,k_i}$ are the corresponding steady state distributions and $i = 1, 2, \cdots, m$.

We should point out that the following matrix Q is also an infinitesimal generator matrix,

$$Q = \begin{pmatrix} Q_{k_1\times k_1} & & & \\ & Q_{k_2\times k_2} & & \\ & & \cdots & \\ & & & Q_{k_m\times k_m} \end{pmatrix}.$$

which associated with

$$\frac{d\left(N(C_{1_1},t),\cdots,N(C_{m_{k_m}},t)\right)}{dt} = \left(N(C_{1_1},t),\cdots,N(C_{m_{k_m}},t)\right) Q. \tag{21}$$

Remark 7. *It has been shown in [13] that for some special examples the equilibrium points of the ODEs coincide the steady state probability distributions of the CTMC underlying the PEPA models. This theorem tells us that this kind coincidence is universal for PEPA models without synchronisation.*

5 Density Evolution as an Alternative Semantics

In this section, we will present an alternative semantics of mapping PEPA models to ODEs. First consider the following PEPA model of interacting processors and resources [4]:

$$\begin{aligned} Processor_0 &\stackrel{def}{=} (task1, r_1).Processor_1 \\ Processor_1 &\stackrel{def}{=} (task2, r_2).Processor_0 \\ Resource_0 &\stackrel{def}{=} (task1, r_1).Resource_1 \\ Resource_1 &\stackrel{def}{=} (reset, s).Resource_0 \\ \underbrace{Processor_0||\cdots||Processor_0}_{N \text{ copies}} &\underset{\{task1\}}{\bowtie} \underbrace{Resource_0||\cdots||Resource_0}_{M \text{ copies}}. \end{aligned}$$

The derived ODEs are as follows [4]:

$$\frac{dN(pro_0,t)}{dt} = -r_1 \min\{N(pro_0,t), N(res_0,t)\} + r_2 N(pro_1,t) \tag{22}$$

$$\frac{dN(pro_1,t)}{dt} = r_1 \min\{N(pro_0,t), N(res_0,t)\} - r_2 N(pro_1,t) \tag{23}$$

$$\frac{dN(res_0,t)}{dt} = -r_1 \min\{N(pro_0,t), N(res_0,t)\} + sN(res_1,t) \tag{24}$$

$$\frac{dN(res_1,t)}{dt} = r_1 \min\{N(pro_0,t), N(res_0,t)\} - sN(res_1,t) \tag{25}$$

where $N(pro_0,t), N(pro_1,t)$ represent the number of processors in the states of $Processor_0$, $Processor_1$ at time t respectively, and similarly $N(res_0,t)$, $N(res_1,t)$ represent the number of resources in the states of $Resource_0$, $Resource_1$ at time t respectively.

According to the theoretical results in the previous sections, we know

$$\frac{N(pro_i,t)}{N} \geq 0, \frac{N(res_i,t)}{M} \geq 0, i = 0, 1$$

and

$$\frac{N(pro_0,t)}{N} + \frac{N(pro_1,t)}{N} = 1, \tag{26}$$

$$\frac{N(res_0,t)}{M} + \frac{N(res_1,t)}{M} = 1. \tag{27}$$

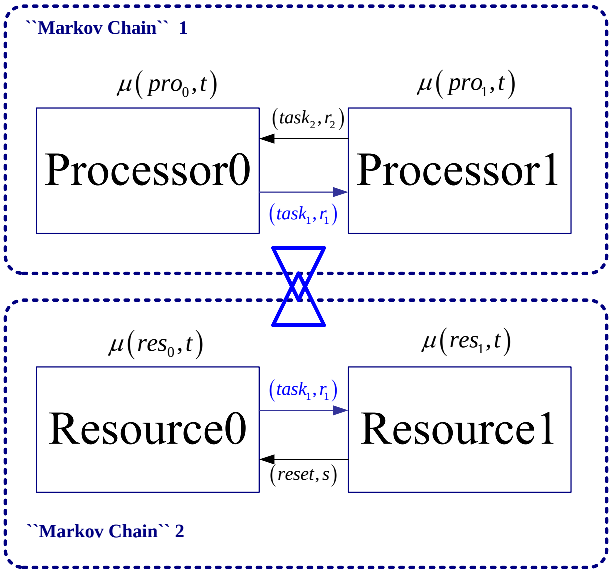


Figure 1: Probability Distribution of Processor and Resource

If the evolution of the realistic systems is essentially determined by the interaction between the densities such as $\frac{N(pro_i,t)}{N}$, $\frac{N(res_i,t)}{M}$ rather than the absolute numbers like $N(pro_i,t)$, $N(res_i,t)$, then we should rewrite the ODEs (22)-(25) as the following

$$\frac{d\mu(pro_0,t)}{dt} = -r_1 \min\{\mu(pro_0,t), \mu(res_0,t)\} + r_2\mu(pro_1,t) \tag{28}$$

$$\frac{d\mu(pro_1,t)}{dt} = r_1 \min\{\mu(pro_0,t), \mu(res_0,t)\} - r_2\mu(pro_1,t) \tag{29}$$

$$\frac{d\mu(res_0,t)}{dt} = -r_1 \min\{\mu(pro_0,t), \mu(res_0,t)\} + s\mu(res_1,t) \tag{30}$$

$$\frac{d\mu(res_1,t)}{dt} = r_1 \min\{\mu(pro_0,t), \mu(res_0,t)\} - s\mu(res_1,t) \tag{31}$$

where

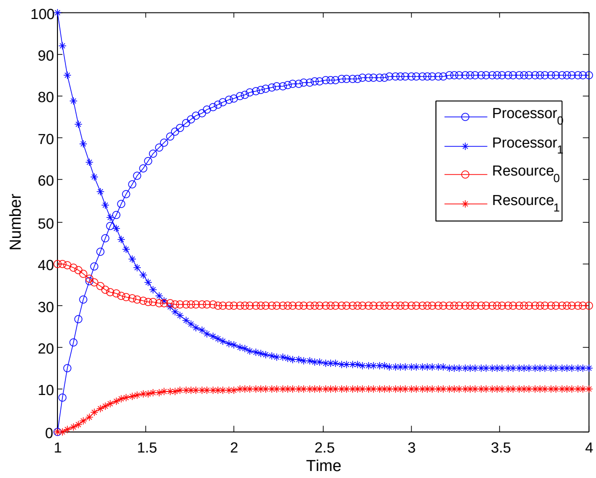
$$\mu(pro_i,t) = \frac{N(pro_i,t)}{N}, \quad \mu(res_i,t) = \frac{N(res_i,t)}{M}, \quad i = 0, 1 \tag{32}$$

represent the probability distribution of a Processor and a Resource at time t respectively, see Figure 1.

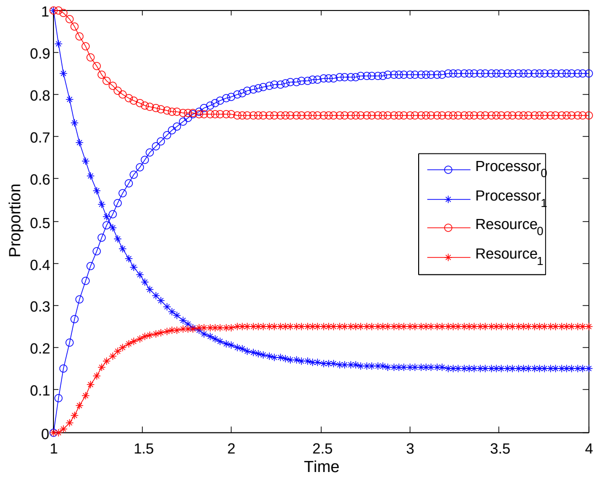
It is obvious that (22)-(25) are not equivalent to (28)-(31) except for the case of $N = M$.

Set $N = 100, M = 30, r_1 = 1, r_2 = 2, s = 3, N(pro_0, 0) = N(res_1, 0) = 0$, then use Euler method to numerically solve (22)-(25) and (28)-(31) respectively. Figure 2 and Figure 3 show the different results.

In order to illustrate the density evolution (28)-(31) has realistic sense, we consider a fancied scenario. In a very rainy city everyone needs an umbrella whenever he/she goes outdoors. So every building has a supply of umbrellas and people just collect or leave them at the front door to the building. When it has been used an umbrella needs some time to dry out before it is ready for using again. We can construct



(a) Absolute Number



(b) $\frac{N(pro_i,t)}{N}$, $\frac{N(res_i,t)}{M}$, $i = 0, 1$

Figure 2: Numerical Solutions of ODEs (22)-(25)

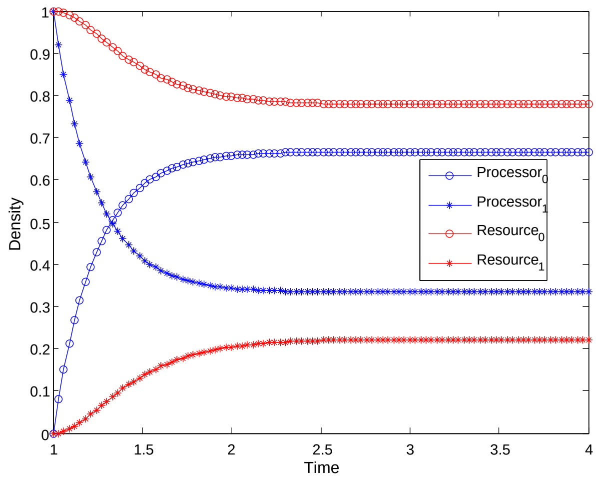


Figure 3: Density Evolution of Processors and Resources (Numerical Solutions of ODEs (28)-(31))

a simple model of umbrellas and workers who have to move between buildings in the course of their working day. The details have been described in the following PEPA model.

$$\begin{aligned}
Umbrella &\stackrel{def}{=} (use, r_1).Wet_Umbrella \\
Wet_Umbrella &\stackrel{def}{=} (dry, r_2).Umbrella \\
User &\stackrel{def}{=} (use, r_1).Worker \\
Worker &\stackrel{def}{=} (work, t).User
\end{aligned}$$

$$\underbrace{Umbrella||\cdots||Umbrella}_{N \text{ copies}} \underset{\{use\}}{\bowtie} \underbrace{User||\cdots||User}_{M \text{ copies}}.$$

Then a density measure could be what percentage of umbrellas are in use. Thus it is natural for us to prefer the ODEs forms (28)-(31).

The new semantics is based on the original one. For example, for a general PEPA model, after we deriving the ODEs (1) by the original semantics [4], we immediately turn them into the following forms, which describe the density evolution of the system.

$$\begin{aligned}
\frac{d\mu\left(C_{i_j}, t\right)}{dt} = &- \sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)} \left\{\mu\left(C_{k_l}, t\right)\right\} \\
&+ \sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in En(\alpha,r)} \left\{\mu\left(C_{k_l}, t\right)\right\}
\end{aligned} \tag{33}$$

6 Conclusions

This paper is focused on the theoretical aspects of the ODEs derived from PEPA models. We show the existence, uniqueness and boundedness of solutions of the ODEs. We also present the convergence of the solutions for non-synchronised cases as well as its relationship to the steady state distributions of some Markov chains. Based on the theoretical results, we propose an alternative semantics of mapping PEPA models to ODEs. Some other theoretical problems (e.g., the convergence problem of the ODEs from general PEPA models) on the ODEs are still under investigation.

When the rates of change within the system model are generalised to allow activity rates to be governed by probability distributions rather than being deterministic, or more uncertainty is involved, the evolution of the system can be described by a set of random or stochastic differential equations [4]. So, the further work involves developing a semantics which should encompass both a Markovian and SDEs interpretation, and even implementing the semantics to provide a tool which allows models to be automatically used to derive the underlying mathematical structures for evaluation. Moreover, analogously to the previous work with ODEs, we should determine the characteristics of the SDEs and their relationships with the CTMC.

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Appendix A

Theorem A1: Fundamental Inequality ([14], page 14) If $\mathbf{x}^{'} = \mathbf{f}(t, \mathbf{x})$ is defined on a set U in $\mathbb{R} \times \mathbb{R}^n$ with Lipschitz condition

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| < K\|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $(t, \mathbf{x}_1)$ and $(t, \mathbf{x}_2)$ on U, and if for $\epsilon_i, \delta \in \mathbb{R}$, and $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are two continuous, piecewise differentiable functions on U into $\mathbb{R}^n$ with

$$\|\mathbf{u}_i^{'}(t) - \mathbf{f}(t, \mathbf{u}_i(t))\| \leq \epsilon_i,$$

and

$$\|\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)\| \le \delta,$$

then

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \le \delta e^{K(t-t_0)} + \left(\frac{\epsilon_1 + \epsilon_2}{K}\right)\left(e^{K(t-t_0)} - 1\right).$$

Theorem A2: Existence and Uniqueness([14], page 14) If $\mathbf{x}^{'} = \mathbf{f}(t, \mathbf{x})$ is defined on a set U in $\mathbb{R} \times \mathbb{R}^n$ with Lipschitz condition

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| < K\|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $(t, \mathbf{x}_1)$ and $(t, \mathbf{x}_2)$ on U, then there exists a unique solution $\mathbf{x} = \mathbf{u}(t)$ for a given set of initial condition $\mathbf{x}(t_0)$

Appendix B

We show the "min" is a Lipschitz function. We only prove the two-dimensional case:

$$\|\min(x_1, x_2) - \min(y_1, y_2)\| \le K\|(x_1, x_2) - (y_1, y_2)\| \stackrel{def}{=} K(|x_1 - y_1| + |x_2 - y_2|).$$

Proof. Noticing $\min(a, b) = \frac{a+b-|a-b|}{2}$, then

$$\begin{aligned} \min(x_1, x_2) - \min(y_1, x_2) &= \frac{x_1 - y_1}{2} + \frac{|x_1 - x_2| - |y_1 - x_2|}{2} \\ &\le \frac{1}{2}|x_1 - y_1| + \frac{1}{2}|(x_1 - x_2) - (y_1 - x_2)| \\ &= |x_1 - y_1|. \end{aligned}$$

Hence we have

$$\begin{aligned} |\min(x_1, x_2) - \min(y_1, y_2)| &= |\min(x_1, x_2) - \min(y_1, x_2) + \min(y_1, x_2) - \min(y_1, y_2)| \\ &\le |\min(x_1, x_2) - \min(y_1, x_2)| + |\min(y_1, x_2) - \min(y_1, y_2)| \\ &\le |x_1 - y_1| + |x_2 - y_2|. \end{aligned}$$

□