

Everything you wanted to know about RCAT but were afraid to ask

Maria Grazia Vigliotti

joint work with Peter Harrison

Plan of the talk

- Reversibility and reversed processes simplify the study of finding steady state distributions.
- Reversibility It has applications in biology and complex systems.
- We will study reversed process in the context of CTMC.
- Stochastic process algebra embeds CTMC

What does it mean to have reversible/reversed processes in process algebra?

Reversible processes

- If $\{X(t_1), X(t_2), \dots, X(t_n)\}$ has the same distribution as $\{X(\tau - t_1), X(\tau - t_2), \dots, X(\tau - t_n)\}$ then the process is called *reversible*.
- Reversible processes are stationary.
- Reversible processes and satisfy the *detailed balance equation*

$$\pi(i)q_{ij} = \pi(j)q_{ji}$$

(viceversa)

- Kolmogorov's criteria Markov process is reversible if and only if

$$q_{j_0 j_1} q_{j_1 j_2} \cdots q_{j_n j_0} = q_{j_n j_0} q_{j_n j_{n-1}} \cdots q_{j_0 j_1}$$

for any finite sequence of states.

PEPA

$$P ::= (a, \lambda).P \mid P + P \mid A$$

$$N ::= P \mid N \underset{L}{\bowtie} N$$

$$\frac{}{(a, \lambda).P \xrightarrow{(a, \lambda)} P}$$

$$\frac{P \xrightarrow{(a, \lambda)} P'}{P + Q \xrightarrow{(a, \lambda)} P'}$$

$$\frac{P \xrightarrow{(a, \lambda)} P'}{A \xrightarrow{(a, \lambda)} P'} (A = P)$$

$$\frac{P \xrightarrow{(a, \lambda)} P'}{P \underset{L}{\bowtie} Q \xrightarrow{(a, \lambda)} P' \underset{L}{\bowtie} Q}$$

$$\frac{P \xrightarrow{(a, \lambda)} P' \quad Q \xrightarrow{(a, \top)} Q'}{P \underset{L}{\bowtie} Q \xrightarrow{(a, \lambda)} P' \underset{L}{\bowtie} Q'}$$

Markov Chains via Process algebra

● Rate from state to state: $q_{P,P'} \stackrel{df}{=} \sum_{P \xrightarrow{(c,\lambda)} P'} \hat{\lambda}$ and rate making

action a $q_{P,a,P'} \stackrel{df}{=} \sum_{c \in Act, P \xrightarrow{(a,\lambda)} P'} \hat{\lambda}$

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- Exit rate: $\mathbf{E}(P) \stackrel{df}{=} \sum_{P' \in ds(P)} \mathbf{q}_{P,P'}$;
- Prob. changing state: $\mathbf{p}(P,P') \stackrel{df}{=} \mathbf{q}_{P,P'} / \mathbf{E}(P)$, if $\mathbf{E}(s)$ is defined. A process P is *irreducible* if and only if for some n and P if $P \xrightarrow{(a_1,r_1)} \xrightarrow{(a_2,r_2)} \dots \xrightarrow{(a_n,r_n)} P'$ then for some m $P' \xrightarrow{(b_1,t_1)} \xrightarrow{(b_2,t_2)} \dots \xrightarrow{(b_m,t_m)} P$.

Reversible processes in PEPA

Let's consider the birth and death processes:

$$\begin{aligned}P_0 &= (a, \lambda).P_1 \\P_n &= (a, \lambda).P_{n+1} + (s, \mu).P_{n-1} \quad n = 3\end{aligned}$$

This process is reversible:

$$\begin{aligned}q_{P_0, P_1} q_{P_1, P_0} &= q_{P_0, P_1} q_{P_1, P_0} \\q_{P_0, P_1} q_{P_1, P_2} q_{P_2, P_1} q_{P_1, P_0} &= q_{P_0, P_1} q_{P_1, P_2} q_{P_2, P_1} q_{P_1, P_0} \\&\vdots = \vdots\end{aligned}$$

Reversed processes

- Reversibility is a very strong condition.
- Most Markov processes (thus PEPA process) are not reversible, that is not to say that the *reversed* process has not any interest.
- With the reversed process it is easy to calculate steady state distribution.

Markov process with generator matrix \mathbf{Q} and the reversed one with generator matrix $\overline{\mathbf{Q}}$ if and only if

$$\pi(j)\overline{q}_{ji} = \pi(i)q_{ij}$$

Reversed processes in PEPA

$$P_0 = (a, \lambda).P_1$$

$$P_1 = (b, \mu).P_2$$

$$P_2 = (c, \gamma).P_0$$

$$\overline{P_0} = (\overline{c}, \overline{\gamma}).\overline{P_2}$$

$$\overline{P_2} = (\overline{b}, \overline{\mu}).\overline{P_1}$$

$$\overline{P_1} = (\overline{a}, \overline{\lambda}).\overline{P_0}$$

What are the reversed rates $\overline{\gamma}, \overline{\mu}, \overline{\lambda}$?

Reversed rates

Reversed rates can be calculate using Kolmogorov's generalised criteria: A Markov process with the generator matrix Q and the reversed process has the generator matrix \bar{Q} if and only if for every finite sequence of states

$$q_i = \bar{q}_i$$
$$q_{j_0 j_1} q_{j_1 j_2} \cdots q_{j_n j_0} = \bar{q}_{j_0 j_n} \bar{q}_{j_{n-1} j_n} \cdots \bar{q}_{j_1 j_0}$$

In the previous case we have that $\bar{\gamma} = \lambda, \bar{\lambda} = \mu, \bar{\mu} = \gamma$.

More on reversed rates

$$P_0 = (a, \lambda).P_1 + (d, \delta).P_1$$

$$P_1 = (b, \mu).P_2$$

$$P_2 = (c, \gamma).P_0$$

$$\overline{P}_0 = (\overline{c}, \overline{\gamma}).\overline{P}_2$$

$$\overline{P}_2 = (\overline{b}, \overline{\mu}).\overline{P}_1$$

$$\overline{P}_1 = (\overline{a}, \overline{\lambda}).\overline{P}_0 + (\overline{d}, \overline{\delta}).\overline{P}_0$$

Now what is the reversed rates of $\overline{\delta}, \overline{\lambda}$? We have by Kolmogorov criteria the following set of equations:

$$\delta + \lambda = \overline{\gamma} \quad \mu = \overline{\delta} + \overline{\lambda} \quad \gamma = \overline{\mu}$$

More on reversed rates

$$\overline{P}_0 = (\overline{c}, \delta + \mu) \cdot \overline{P}_2$$

$$\overline{P}_2 = (\overline{b}, \gamma) \cdot \overline{P}_1$$

$$\overline{P}_1 = (\overline{a}, \frac{\lambda}{\lambda + \delta} \mu) \cdot \overline{P}_0 + (\overline{d}, \frac{\delta}{\lambda + \delta} \mu) \cdot \overline{P}_0$$

One can generalise this definition and had a way of obtaining the reversed rate in case of multiple actions that is

$$\overline{\mathbf{q}}_{P,a,P'} = \frac{\mathbf{q}_{P,a,P'}}{\mathbf{q}_{P,P'}} \overline{\mathbf{q}}_{P,P'}$$

Reversed PEPA symbolic processes

The reversed process \overline{P} is defined as follows:

$$\begin{aligned} \overline{T} &\stackrel{df}{=} \sum_{i: S_i \xrightarrow{(a_i, r_i)} T} (\overline{a_i}, \overline{r_i}) \cdot \overline{S_i} \\ \overline{N_1 \bowtie_L N_2} &\stackrel{df}{=} \overline{N_1} \bowtie_{\overline{L}} \overline{N_2} \end{aligned}$$

The real process is the one where reversed rates $\overline{\lambda} = r$ with $r \in \mathbb{R}$

RCAT

$P \bowtie_L Q$ be the PEPA process such that the processes P and Q are irreducible. Given that:

- every passive action type in $\mathcal{P}_P(L)$ or $\mathcal{P}_Q(L)$ is always enabled in P or in Q .
- every reversed action of an active action type in $\mathcal{A}_P(L)$ or $\mathcal{A}_Q(L)$ is always enabled in \bar{P} or in \bar{Q} .
- for all $a \in \mathcal{A}_P(L)$ $\mathbf{q}_{P,a,P'} = \mathbf{q}_{\bar{P},a,\bar{P}'}$ and for all $a \in \mathcal{A}_Q(L)$ $\mathbf{q}_{Q,a,Q'} = \mathbf{q}_{\bar{Q},a,\bar{Q}'}$

then the reversed agent $\overline{P \bowtie_L Q}$, is:

$$R^* \bowtie_{\bar{L}} S^*$$

where:

RCAT (Cont.)

$$\begin{aligned}R^* &= \bar{R}\{(\bar{a}, \bar{p}_a) \leftarrow (\bar{a}, \top) \mid a \in \mathcal{A}_P(L)\} \\S^* &= \bar{S}\{(\bar{a}, \bar{q}_a) \leftarrow (\bar{a}, \top) \mid a \in \mathcal{A}_Q(L)\} \\R &= P\{(a, \top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_P(L)\} \\S &= Q\{(a, \top) \leftarrow (a, x_a) \mid a \in \mathcal{P}_Q(L)\}.\end{aligned}$$

The symbolic rates $\{x_a\}$ are given by the solution of the following set of equation:

$$\begin{aligned}\top_a &= \bar{q}_a && \text{if } a \in \mathcal{P}_P(L) \\ \top_a &= \bar{p}_a && \text{if } a \in \mathcal{P}_Q(L)\end{aligned}$$

and \bar{p}_a, \bar{q}_a are the symbolic actions of type a in \bar{P} and \bar{Q} respectively.

Application to simple process

$$A = (a, \lambda).A$$

$$B = (a, \top).B$$

$$C = (c, \gamma).(d.\delta).C$$

$$A \bowtie_a B \bowtie C$$

What is

$$\overline{A \bowtie_a B \bowtie C}?$$

$$\overline{A \bowtie_a B \bowtie C} = \overline{A} \bowtie_a \overline{B} \bowtie \overline{C} \\ A \bowtie_a B \bowtie \overline{C}$$

Cont.

where $\bar{C} = (\bar{d}.\bar{\delta}).(\bar{c},\bar{\gamma}).\bar{C}$ by RCAT $x_a = \lambda = \bar{\lambda}$ thus we have

$$\bar{A} = (a, \top).\bar{A}$$

$$\bar{B} = (a, \lambda).\bar{B}$$

$$\bar{C} = (\bar{d}.\bar{\delta}).(\bar{c},\bar{\gamma}).\bar{C}$$

What if there is more than one action moving from one state to another?

Networks of queues

$$\begin{aligned}N_{1,0} &= (e_1, \lambda_1).N_{1,1} + (a_{2,1}, \top_{2,1}).N_{1,1} \\N_{1,n} &= (e_1, \lambda_1).N_{1,n+1} + (a_{2,1}, \top_{2,1}).N_{1,n+1} \\&\quad (a_{1,2}, p_{1,2}\mu_1).N_{1,n-1} + \\&\quad (d_1, (1 - p_{1,2})\mu_1).N_{1,n-1}\end{aligned}$$

$$\begin{aligned}N_{2,0} &= (e_2, \lambda_2).N_{2,1} + (a_{1,2}, \top_{1,2}).N_{2,1} \\N_{2,n} &= (e_2, \lambda_2).N_{2,n+1} + (a_{1,2}, \top_{1,2}).N_{2,n+1} \\&\quad (a_{2,1}, p_{2,1}\mu_2).N_{2,n-1} + \\&\quad (d_2, (1 - p_{2,1})\mu_2).N_{2,n-1}\end{aligned}$$

$$N_{1,0} \begin{array}{c} \text{---} \times \text{---} \\ \{a_{2,1}, a_{1,2}\} \end{array} N_{2,0}$$

Reversed the queues

$$\begin{aligned}N_{1,0} &= (e_1, \lambda_1).N_{1,1} + (a_{2,1}, x_{2,1}).N_{1,1} \\N_{1,n} &= (e_1, \lambda_1).N_{1,n+1} + (a_{2,1}, x_{2,1}).N_{1,n+1} \\&\quad (a_{1,2}, p_{1,2}\mu_1).N_{1,n-1} + \\&\quad (d_1, (1 - p_{1,2})\mu_1).N_{1,n-1}\end{aligned}$$

Reverse the queue:

$$\begin{aligned}\overline{N_{1,0}} &= (\overline{d_1}, \overline{1 - p_{1,2}}\mu_1).\overline{N_{1,1}} + (\overline{a_{2,1}}, \overline{x_{2,1}}).\overline{N_{1,1}} \\ \overline{N_{1,n}} &= \dots\end{aligned}$$

The rates

By condition 3 we have that:

$$\begin{aligned}x_{2,1} &= \mathbf{q}_{N_{2,n}, N_{2,n-1}} = \overline{p_{12}\mu_1} = \frac{p_{21}\mu_2}{\mu_2} (\lambda_2 + x_{1,2}) = p_2 1v_2 \\x_{1,2} &= \mathbf{q}_{N_{1,n}, N_{1,n-1}} = p_{12}(\lambda_1 + x_{2,1})\end{aligned}$$

We assume that **the following rates are known:**

$$\begin{aligned}\overline{\mathbf{q}_{N_{1,n}, N_{1,n-1}}} &= \mu_1 \\ \overline{\mathbf{q}_{N_{1,n}, N_{1,n+1}}} &= v_1 \\ \overline{\mathbf{q}_{N_{2,n}, N_{2,n-1}}} &= \mu_2 \\ \overline{\mathbf{q}_{N_{1,n}, N_{2,n+1}}} &= v_2 \\ \overline{d_1} &= e_1 \\ \overline{a_1} &= a_1 \\ \overline{e_1} &= d_1 \\ \vdots &= \vdots\end{aligned}$$

So.....

$$\overline{N_{1,n}} = \left(e_1, \frac{\lambda_1}{v_1} \mu_1\right) \cdot \overline{N_{1,n-1}} + \left(a_2 1, \frac{x_{21}}{v_1} \mu_1\right) \cdot \overline{N_{1,N-1}}$$
$$\left(d_1, (1 - p_{21})v_1\right) \cdot N_{n+1} + \left(a_2, \top\right) \cdot N_{1,n+1}$$

The same for the node $N_{2,n}$.

Conclusions

- RCAT gives a natural way to find the reversed PEPA process.
- The reversed rates have to be KNOWN.
- RCAT can be applied to other kinds of process algebra.
- RCAT can be applied to process algebra with immediate actions without any significant modification: it suffice to make vanish the immediate actions.



Thanks