On ODEs From PEPA Models

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July 25, 2007

Outlines

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- Instead of calculating the probability distribution, choosing a more abstract state representation in terms of state variables, quantifying the types of behaviour evident in the model.
- Assuming that these state variables are subject to continuous rather than discrete change.

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- Assuming that these state variables are subject to continuous rather than discrete change.

ODEs from PEPA Models

Jane used a new state representation—numerical vector form in [2]. Let $N(C_{i_j}, t)$ denote the jth entry of the ith subvector at time t, i.e. the number of instances of the jth local derivative of sequential component C_i. Then,

$$\frac{dN\left(C_{i_j}, t\right)}{dt} = -\sum_{k: C_{i_j} \xrightarrow{(a,\cdot)} C_{ik}} \text{exit rate from } C_{i_j} \text{ to } C_{i_k} + \sum_{k: C_{i_k} \xrightarrow{(a,\cdot)} C_{ij}} \text{entry rate into } C_{i_j} \text{ from } C_{i_k}$$

The formal form of that equation is as the following [2]

$$\frac{dN\left(C_{i_j},t\right)}{dt} = -\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)} \left\{N\left(C_{k_l},t\right)\right\} + \sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in En(\alpha,r)} \left\{N\left(C_{k_l},t\right)\right\} \tag{1}$$

Example

$$Processor_0 \stackrel{\text{def}}{=} (task1, r_1).Processor_1$$

$$Processor_1 \stackrel{\text{def}}{=} (task2, r_2).Processor_0$$

$$Resource_0 \stackrel{\text{def}}{=} (task1, r_1).Resource_1$$

$$Resource_1 \stackrel{\text{def}}{=} (reset, s).Resource_0$$

$$\underbrace{Processor_0 || \cdots || Processor_0}_{N\ \ copies} \underset{\{task1\}}{\bowtie} \underbrace{Resource_0 || \cdots || Resource_0}_{M\ \ copies}.$$

The derived ODEs are as follows [2]:

$$
\begin{aligned}
\frac{dN(pro_0,t)}{dt} &= -r_1 \min\{N(pro_0,t), N(res_0,t)\} + r_2 N(pro_1,t) \\
\frac{dN(pro_1,t)}{dt} &= r_1 \min\{N(pro_0,t), N(res_0,t)\} - r_2 N(pro_1,t) \\
\frac{dN(res_0,t)}{dt} &= -r_1 \min\{N(pro_0,t), N(res_0,t)\} + s N(res_1,t) \\
\frac{dN(res_1,t)}{dt} &= r_1 \min\{N(pro_0,t), N(res_0,t)\} - s N(res_1,t)
\end{aligned}
$$

where $N(pro_0,t), N(pro_1,t)$, $N(res_0,t)$, $N(res_1,t)$ represent the number of processors and $Resource_0$ in the states of $Processor_0$, $Processor_1$, $Resource_0$, $Resource_1$ at time t respectively.

Features of ODEs

Proposition 1

$$\frac{dN\left(C_{i_j},t\right)}{dt} = -\sum_{k:C_{i_j}\xrightarrow{(a,\cdot)}C_{ik}} \text{exit rate from } C_{i_j} \text{ to } C_{i_k} + \sum_{k:C_{i_k}\xrightarrow{(a,\cdot)}C_{ij}} \text{entry rate into } C_{i_j} \text{ from } C_{i_k}$$

Notice such an important fact: the sum of all exit activities rates equal to the sum of all entry activities rates for each type of component at any time, since the system is closed and there is no exchange with outside environment.

Features of ODEs

Proposition 1

Proposition

The ODEs satisfy Conservation Law, i.e., the number of each kind of component keeps unchanged all the time.
In other words, for all i,

$$\sum_j \frac{dN\left(C_{i_j}, t\right)}{dt} = 0, \quad \forall t, \tag{2}$$

or

$$\sum_j N\left(C_{i_j}, t\right) = \sum_j N\left(C_{i_j}, 0\right) = N(C_i), \quad \forall t. \tag{3}$$

Another important fact to note is: the “exit rates” of C_{i_j},

$$\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)} \left\{N\left(C_{k_l},t\right)\right\},$$

or

$$\sum_{k: C_{i_j} \xrightarrow{(a,\cdot)} C_{ik}} \text{exit rate from } C_{i_j} \text{ to } C_{i_k}$$

is related to the number of C_{i_j}. According to the mapping semantics, the exit activities of C_{i_j} depend on either C_{i_j} itself or the synchronisation in which C_{i_j} takes part.

This implies

$$\min_{C_{k_l} \in Ex(\alpha,r)} \{N(C_{k_l}, t)\} \leq N(C_{i_j}, t).$$

So we have

Proposition

For all C_{i_j},

$$\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l} \in Ex(\alpha,r)} \{N(C_{k_l}, t)\} \leq \sum_{(\alpha,r)\in Ex(\alpha,r)} r \times N(C_{i_j}, t)$$

Main Theoretical Results

Existence, Uniqueness and Boundness of Solutions

$$\frac{dN\left(C_{i_j},t\right)}{dt} = -\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in Ex(\alpha,r)} \left\{N\left(C_{k_l},t\right)\right\} + \sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r \times \min_{C_{k_l}\in En(\alpha,r)} \left\{N\left(C_{k_l},t\right)\right\}$$

The solutions of above ODEs are not only existent and unique, but also also bounded.

Main Theoretical Results

Theorem 2: Boundness of Solutions

Theorem

Suppose $N(C_{i_j}, t)$ *satisfy above ODEs with nonnegative initial values, then*

$$0 \le N(C_{i_j}, t) \le N(C_i) = \text{total number of component } C_i, \quad \forall t.$$

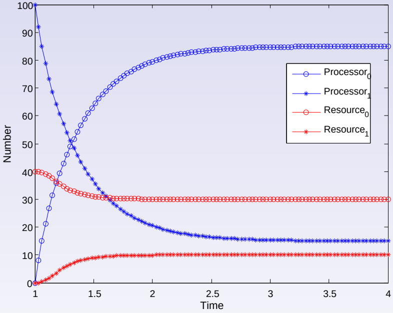
Moreover, if the initial values are positive, then the solutions are positive all the time, i.e.,

$$0 < N(C_{i_j}, t) \le N(C_i) = \text{total number of component } C_i, \quad \forall t.$$

Recall the tiny example. The derived ODEs:

$$
\begin{aligned}
\frac{dN(pro_0,t)}{dt} &= -r_1 \min\{N(pro_0,t), N(res_0,t)\} + r_2 N(pro_1,t) \\
\frac{dN(pro_1,t)}{dt} &= r_1 \min\{N(pro_0,t), N(res_0,t)\} - r_2 N(pro_1,t) \\
\frac{dN(res_0,t)}{dt} &= -r_1 \min\{N(pro_0,t), N(res_0,t)\} + s N(res_1,t) \\
\frac{dN(res_1,t)}{dt} &= r_1 \min\{N(pro_0,t), N(res_0,t)\} - s N(res_1,t)
\end{aligned}
$$

Given the parameters and the initial values: $r_1 = 1, r_2 = 2, s = 3$, $N(pro_0, 0) = 0, N(pro_1, 0) = 100$, $N(res_0, 0) = 0, N(res_1, 0) = 40$. The numerical solutions are as following



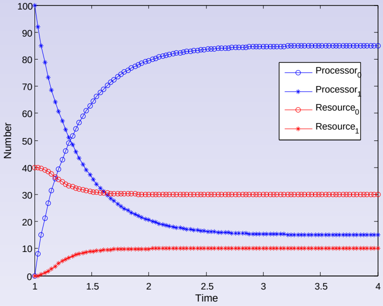
$$0 < N(pro_0, t), N(pro_1, t) < 100,$$

$$0 < N(res_0, t), N(res_1, t) < 40,$$

which are consistent with the Theorem 2.

Main Theoretical Results

Convergence of Solutions



- The solutions in above figure converge to finite limits as time goes to infinity.
- It is true, for this tiny example, we can prove it.

Main Theoretical Results

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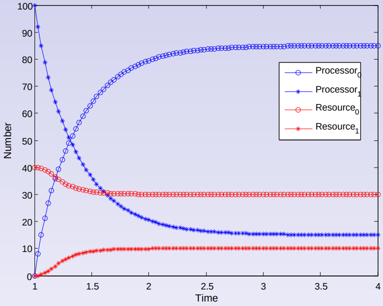

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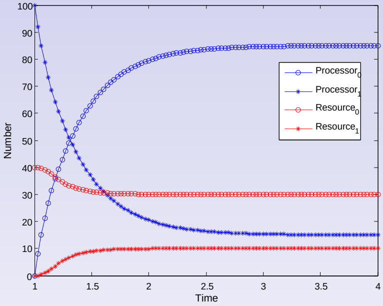
- Unfortunately, for general cases, we have no theoretical results yet.

Main Theoretical Results
Convergence of Solutions



- The solutions in above figure converge to finite limits as time goes to infinity.
- It is true, for this tiny example, we can prove it.
- Unfortunately, for general cases, we have no theoretical results yet.
- But for non-synchronised PEPA models, we have $\cdots\cdots$

If there is no synchronisation in the system, the ODEs will become

$$\begin{aligned}\frac{dN\left(C_{i_j},t\right)}{dt} = & -\sum_{(\alpha,r)\in Ex\left(C_{i_j}\right)} r\times N\left(C_{i_j},t\right) \\ & +\sum_{(\alpha,r)\in En\left(C_{i_j}\right)} r\times N\left(C_{i_j},t\right)\end{aligned} \tag{4}$$

Main Theoretical Results

Theorem 3: Convergence of Non-synchronised Case

Theorem

Suppose $N\left(C_{i_j}, t\right)$ satisfy (4), then for any nonnegative initial values, we have

$$\lim_{t\to\infty} N(C_{i_j}, t) = N(C_{i_j}, \infty) = N(C_i)\pi(C_{i_j}, \infty), \quad \forall i, j. \quad (5)$$

where $\pi(C_{i,j}, \infty)$ are the corresponding steady state distribution (of the Markov chain underlying the one-copy-component PEPA model).

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Remark

It has been shown in [1] that for some special examples the equilibrium points of the ODEs coincide the steady state probability distributions. This theorem tells us that this kind coincidence is universal for PEPA models without synchronisation.

Suppose Q is the infinitesimal matrix of a CTMC, the probability distribution $\pi(t)$ satisfies

$$\frac{d\pi(t)}{dt} = \pi(t)Q$$

To my best knowledge, even for the linear ODEs, no one has shown the proof of the convergence of $\pi(t)$ based on the ODEs' theory (of course except for some special cases). The general proof of the convergence is just based on a probabilistic method.

The derived ODEs from PEPA models are usually nonlinear which is due to synchronisation. We also couldn't totally rely on probabilistic methods since the relationship between derived ODEs and the CTMCs has not been very well known. So, We have to face the difficulties in dealing with the nonlinearity.

Conclusions

- The existence, uniqueness and boundedness of solutions of the derived ODEs;
- Convergence of the solutions for non-synchronised cases as well as its relationship to the steady state distributions of some Markov chains;
- Difficulty for synchronised models: synchronisation results in nonlinearity.

Acknowledgement

The work reported in this paper has formed part of the Ubiquitous Services Core Research Programme of the Virtual Centre of Excellence in Mobile & Personal Communications, Mobile VCE, www.mobilevce.com. This research has been funded by the DTI-led Technology Programme and by the Industrial Companies who are Members of Mobile VCE. Fully detailed technical reports on this research are available to Industrial Members of Mobile VCE. J. Hillston is also supported by EPSRC Advanced Research Fellowship EP/c543696/01 and EU FET-IST Global Computing 2 project SENSORIA ("Software Engineering for Service-Oriented Overlay Computers" (IST-3-016004-IP-09)). J. Ding acknowledges the support of the Scottish Funding Council for the Joint Research Institute with the Heriot-Watt University which is a part of the Edinburgh Research Partnership.

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Thank you!

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