A tandem network of fluid queues with on-off arrivals

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Why fluid queues?

- exactly describe flow-systems with *continuous state*
  - e.g. volumes in literal fluid flows
  - applications in dynamics of *real* fluids
- Approximation in *discrete state systems* for *traffic flows*
  - state spaces in discrete systems become prohibitively large as complexity increases
  - more concise to represent state by a real number and transitions by derivatives
  - often a much simpler solution, either analytical or by stochastic simulation

Fluid limits

- As discrete rates tend to infinity, system's behaviour often approaches a fluid model
  - depends on the limiting regime, e.g. relative rates of increase between arrival rates and service rates
  - see the book by Whitt
- Used a lot ever since the 70s, e.g. papers by Gaver, Kobayashi, Gelenbe, Mitra, etc
- But in practice, not *necessary* for a fluid model to be the limit of any known discrete system

Single fluid queue

- Standard result, but solution usually ‘guessed’
- We derive the differential equation informally and obtain a solution constructively
  - simplified since the fluid model is *first order*
  - inclusion of random ‘noise’ would lead to a harder second-order equation using Ito calculus etc.
- Idea is that the approach then extends to simple *networks* and non-queueing systems

Problem definition

- n phases in CTMC, with generator matrix $Q = (q_{ij} \mid 1 \le i, j \le n)$ and equilibrium probabilities $\vec{\pi}$
- Arrival rate in phase i is λ_i
- Server outputs fluid when its reservoir is non-empty at rate μ
- Rate matrix $R = \mathrm{diag}(r_1, \ldots, r_n)$, where $r_i = \lambda_i - \mu$ for $1 \le i \le n$
- $\vec{F}(x,t) = (F_1(x,t), \ldots, F_n(x,t))$, where

$$F_i(x,t) = \mathbb{P}(N_t = i, X_t \le x)$$

Incremental analysis ($x > 0$)

- Consider the infinitesimal interval $(t, t+h]$ for some small h.

$$\begin{aligned} F_i(x, t+h) &= (1 + q_{ii}h)F_i(x - r_i h, t) \\ &\quad + \sum_{j \neq i} F_j(x, t) q_{ji} h + o(h) \end{aligned}$$

- Terms on the r.h.s. correspond to
  - no phase change w.p. $1 + q_{ii}h$
  - flow of $r_i h$ units of fluid into reservoir
  - phase change $j \rightarrow i$ w.p. $q_{ji}h$

Differential equation

- Rearranging:

$$\frac{F_i(x,t+h)-F_i(x,t)}{h}$$

$$= \quad F_i(x,t)q_{ii} - r_i\frac{\partial F_i(x,t)}{\partial x} + \sum_{j\neq i} F_j(x,t)q_{ji} + O(h)$$

$$= \quad -r_i\frac{\partial F_i(x,t)}{\partial x} + \sum_{j=1}^{n} F_j(x,t)q_{ji} + O(h)$$

Transient and steady states

- In the limit $h \to 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x} R + \vec{F} Q$$

- At equilibrium, when this exists,

$$\vec{F}_x R = \vec{F} Q$$

- Boundary conditions are
  - $F_i(0) = 0$ if $r_i > 0$ (reservoir cannot be empty when there is a positive net input)
  - $\vec{F}(\infty) = \vec{\pi}$ (phase probabilites at *any* fluid level)

Tandem fluid queues

- Two queues with on-off arrivals
  - Generator matrix $Q = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$
  - on-rate λ in phase 1 and 0 in phase 2, no external arrivals at node 2
  - server 1 outputs fluid at rate μ_1 to node 2
  - server 2 outputs fluid at rate μ_2
  - rate matrices $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and $S = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$, where $r_1 = \lambda - \mu_1, r_2 = -\mu_1$ and $s = \mu_1 - \mu_2$.

Incremental analysis ($x, y > 0$)

$$\begin{aligned} F_i(x,y,t+h) &= (1+q_{ii})F_i(x-r_ih, y-s_ih, t) \\ &\quad + \sum_{j\neq i} F_j(x,y,t)q_{ji}h + o(h) \end{aligned}$$

- Terms on the r.h.s. correspond to
  - no phase change w.p. $1 + q_{ii}h$
  - flow of λ or 0 units of fluid into node 1
  - flow of $\mu_1 h$ units of fluid from node 1 to node 2
  - flow of $\mu_2 h$ units of fluid from node 1 to node 2
  - phase change $j \to i$ w.p. $q_{ji}h$

Differential equation

$$\frac{F_i(x,y,t+h)-F_i(x,y,t)}{h} =$$

$$-\,r_i\frac{\partial F_i(x,y,t)}{\partial x} - s_i\frac{\partial F_i(x,y,t)}{\partial y}$$

$$+\sum_{j=1}^{n} F_j(x,y,t)q_{ji} + O(h)$$

so that as $h \longrightarrow 0$

$$\frac{\partial \vec{F}}{\partial t} = -\frac{\partial \vec{F}}{\partial x}R - \frac{\partial \vec{F}}{\partial y}S + \vec{F}Q$$

Steady state

$$\vec{F}_x R + \vec{F}_y S - \vec{F} Q = 0 \qquad (x, y > 0)$$

- But now, when reservoir 1 is empty, there is no input to queue 2
- Boundary equation at $x = 0$:

$$\frac{\partial \vec{F} 0, y)}{\partial x} R + \frac{\partial \vec{F}(0, y)}{\partial y} S' - \vec{F}(0, y) Q = 0$$

  where $S' = -\mu_2 I$, $F_1(0, y) = 0$ for all $y \geq 0$ and $F_1(x, 0) = F_2(x, 0) = 0$ for $x > 0$.

- Boundary condition at infinity $\vec{F}(\infty, \infty) = \vec{\pi}$.

Solution for the single fluid queue

- Laplace transform $\vec{F}^*(\theta) = \int_{0^-}^{\infty} e^{-\theta x}\vec{F}(x)dx$
- Multiply o.d.e. by $e^{-\theta x}$ and integrate 0^+ to ∞

$$\vec{F}^*(\theta)(\theta R - Q) = \vec{F}(0)R$$

- Multiply by θ and set $\theta = 0$ gives $\vec{f}^*(0)Q = 0$ so $\vec{F}(\infty)Q = 0$, cf. $\vec{F}(\infty) = \vec{\pi}$

$$\vec{F}^*(\theta) = \frac{\vec{F}(0)R}{\theta(r_1 r_2\theta + r_1 b + r_2 a)} \begin{pmatrix} r_2\theta + b & a \\ b & r_1\theta + a \end{pmatrix}$$

Solution (2)

- Multiplying by θ, we get

$$\vec{f}^*(\theta) = \frac{r_2 F_2(0)}{r_1 r_2 \theta + r_1 b + r_2 a}(b, r_1\theta + a)$$

- At $\theta = 0$,

$$\vec{\pi} = \frac{r_2 F_2(0)}{r_1 b + r_2 a}(b, a)$$

- But $\vec{\pi} = \frac{(b,a)}{a+b}$ so (after simplification)

$$F_2(0) = 1 - \pi_1 \lambda / \mu_1$$

- So $\pi_1 \lambda / \mu_1$ is the utilisation, as expected.

Solution (3)

- Full solution is:

$$\vec{f}^*(\theta) = (0, F_2(0)) + \frac{\pi_1 \alpha}{\theta + \alpha}(1, (\lambda - \mu_1)/\mu_1)$$

where $\alpha = -\frac{r_1 b + r_2 a}{r_1 r_2}$ and $F_2(0) = \frac{-r_1 a}{a+b}$.

- Easily inverted to give

$$\vec{f}(x) = (0, F_2(0))\delta(x) + (1, (\lambda - \mu_1)/\mu_1)\pi_1 \alpha e^{-\alpha x}$$

or

$$\vec{F}(x) = (0, F_2(0)) + (1, (\lambda - \mu_1)/\mu_1)\pi_1 [1 - e^{-\alpha x}]$$

Solution for tandem nodes

- So far, so good, now for the hard part!
- Same method yields

  $$\vec{f}^{*}(\theta,\phi)(\theta R+\phi S-Q) = \phi\vec{f}^{*\cdot}(\theta,0)S+\vec{f}^{\cdot *}(0,\phi)(\theta R+\phi S'')$$

  where

  $S'' = S - S' = \mu_1 I,\quad \vec{f}^{*\cdot}(\theta, y) = \int_0^\infty e^{-\theta x}\frac{\partial\vec{F}}{\partial x}dx$

  and $\vec{f}^{\cdot *}(x,\phi) = \int_0^\infty e^{-\theta y}\frac{\partial\vec{F}}{\partial y}dy$
- Note the marginal distributions

  $\vec{f}^{*\cdot}(0,y) = F_Y(y)$ and $\vec{f}^{\cdot *}(x,0) = F_X(x)$

Marginal distribution at node 1

- When $\phi = 0$,

$$\vec{f}^{*}(\theta,0)(\theta R - Q) = \vec{f}^{\cdot *}(0,0)\theta R$$

i.e.

$$\vec{f}^{*}_{X}(\theta)\theta^{-1}(\theta R - Q) = \vec{F}^{*}_{X}(\theta)(\theta R - Q) = \vec{F}_{X}(0)R$$

- Exactly the single fluid queue result, so

$$\vec{F}_{X}(0) = (0, 1 - \pi_1\lambda/\mu_1)$$

Marginal distribution at node 2

- When $\theta = 0$,

$$\begin{aligned}
&\vec{f}^{*}(0,\phi)(\phi S - Q) \\
&\quad = \phi(\vec{f}^{*\cdot}(0,0)S + \vec{f}^{\cdot *}(0,\phi)S'' - \vec{F}(0,0)S'') \\
&\quad = \phi(\vec{f}_{\mathrm{Y}}(0)(S - S'') + \vec{f}^{\cdot *}(0,\phi)S'') \\
&\quad = \phi(\vec{f}_{\mathrm{Y}}(0)S' + \vec{f}^{\cdot *}(0,\phi)S'')
\end{aligned}$$

since X must be 0 whenever $Y = 0$

Marginal distribution at node 2 (2)

- Proceeding as before,

$$\vec{f}^{*}(0,\phi) = \frac{\vec{f}_Y(0)S' + \vec{f}^{\cdot *}(0,\phi)S''}{s(\phi s + a + b)} \begin{pmatrix} \phi s + b & a \\ b & \phi s + a \end{pmatrix}$$

where $s = \mu_1 - \mu_2$

Marginal distribution at node 2 (3)

- Thus, at $\phi = 0$,

$$\vec{f}^*(0,0) = \vec{\pi} = \frac{\beta(b,a)}{s(a+b)} = \frac{\beta}{s}\vec{\pi}$$

where $\beta = \mu_1 - \pi_1\lambda - \mu_2 f_{Y_2}(0)$

- So we must have $\beta = s$, i.e.

$$\mu_1 - \pi_1\lambda - \mu_2 f_{Y_2}(0) = \mu_1 - \mu_2$$

giving

$$\vec{f}_Y(0) = 1 - \frac{\pi_1\lambda}{\mu_2}$$

Utilisation of node 2

- At equilibrium, arrival rate to node 2 is the same as to node 1, viz. $\pi_1\lambda$
- So the utilisation of the second queue is

$$\pi_1\lambda/\mu_2$$

- Again as expected

Full solution?

- Still does not give the whole solution
- Any ideas gratefully received
- Assuming this can be done .....

What's the point?

- For large networks, make approximating assumptions, e.g. that nodes behave independently
  - Validate the approximations against simulation
  - Validate the simulation against exact results in small networks, e.g. tandem
- Search for separable solutions á là RCAT
  - product-form looks highly unlikely
  - maybe some special structure in the Laplace transform??

Independence approximation

- Need to choose an appropriate arrival process at each node
  - MM fluid arrival process with rate in each phase equal to the total rate from all upstream non-empty nodes
  - transition rate due to an upstream node becoming active is the transition rate at that node from empty to non-empty state
  - in our on-off model, the rate a
  - rates in opposite directions similarly, using upstream nodes, or chosen to match utilisation

Validation

- Paul implemented a simulator in his final year project
- Unfortunately found pretty poor agreement for 2 or 3 nodes in tandem
  - could be due to parameter matching against utilisation rather than upstream node
  - simulator could not be validated against exact results

Plans

- Investigate independence assumption
  - can calculate covariance directly from the Laplace transform ... when we get this to work!
  - certainly can check it via simulation, for arbitrary network topologies
- Find better arrival models for individual nodes
  - more phases in Markov modulated arrival processes
  - non-Markovian arrival processes