Matrix-Vector Splitting for Eigenvector Solution

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Presentation overview

- Preamble
  - An example problem
  - A “nicer” problem
  - How many solutions to a problem?
- Theorem
  - How to find “nicer” problems
  - Upper bound on number of solutions
- What counts as “nice”?
- A “nicer” PageRank problem

A problem → A “nice” problem

- $\vec{x} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 1 & 0 & 0 \end{pmatrix} \vec{x}$

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  - $x_1 = \frac{1}{3}$
  - $x_3 = x_1 = \frac{1}{3}$
  - $x_2 = x_3 + \frac{2}{3} = 1$

How many solutions?

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Theorem

(P) $\mathbf{M} \in \mathbb{R}^{n\times n}$ is irreducible;
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Then,

(C1) There is at most one solution to $\vec{x} = \mathbf{M}\vec{x}$.

(C2) This cannot be other than the *unique* fixed-point $\vec{x}$ (which comprises no zero entries) in

$$\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v},$$ where

$$\max\{|\lambda| : \lambda \text{ is an eigenvalue of } |\mathbf{M} - \mathbf{V}^{(K)}|\} < 1.$$

$$\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v}$$

- For $\vec{v} \in \mathbb{R}^n$, $K \subseteq \{1, 2, \ldots, n\}$ we define $\mathbf{V}^{(K)} \in \mathbb{R}^{n \times n}$ by

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- Both $\begin{cases} \forall i, j (0 \leq V_{ij}^{(K)} \leq M_{ij} \text{ or } M_{ij} \leq V_{ij}^{(K)} \leq 0); \\ \exists i, j (V_{ij}^{(K)} \neq 0). \end{cases}$

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- E.g. $\mathbf{M} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 1 & 0 & 0 \end{pmatrix}$ and $\mathbf{V}^{(K)} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$

“Nicer”?

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ versus } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Sparsity;
Spectral radius;
Interdependency;
Asynchronous solver.

Asynchronous solution for $\vec{x}$

- Class of asynchronous iterations for $\vec{y} = \mathbf{A}\vec{y} + \vec{b}$

$$y_i^{(s+1)} := \begin{cases} \sum_{j=1}^{n} a_{ij} y_j^{(s-d(s,i,j))} + b_i & \text{if} \quad i \in u(s) \\ y_i^{(s)} & \text{if} \quad i \notin u(s) \end{cases}$$

  - u gives entries to be updated at step s.
  - d gives relative age of entries being used in update.

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- Restrictions:
  - Each entry updated infinitely often.
  - $\forall s, i, j \in \mathbb{N}$, $\begin{cases} (s - d(s,i,j)) \to \infty \text{ as } s \to \infty; \\ d(s,i,j) \leq s. \end{cases}$

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- Convergence:
  - $\max\{|\lambda| : \lambda$ is an eigenvalue of $|\mathbf{A}|\} < 1$. ✓
  - $\max\{|\lambda| : \lambda$ is an eigenvalue of $|\mathbf{A}|\} \geq 1$. ✗

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- Convergence:
  - $\vec{x} = (\mathbf{M} - \mathbf{V}^{(K)})\vec{x} + \vec{v}$. √
  - $\vec{x} = \mathbf{M}\vec{x}$. ✗

“Nicer” PageRank equation

- Definitions:
  - $P_{ij} := \begin{cases} \frac{1}{deg(j)} & \text{if there is a hyperlink from } j \text{ to } i \\ 0 & \text{otherwise} \end{cases}$
  - $\vec{v}$ is a probability vector
  - $D_{ij} := \begin{cases} v_i & \text{if deg}(j)\text{=0} \\ 0 & \text{otherwise} \end{cases}$
  - $E_{ij} := v_i$
  - $0 < \alpha < 1$

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  - $0 < \alpha < 1$
- PageRank Equation:
  - $\vec{x} = (\alpha\mathbf{P} + \alpha\mathbf{D} + (1-\alpha)\mathbf{E})\vec{x}$

“Nicer” PageRank equation

- *Removing* $\mathbf{D}$, $\mathbf{E}$:

$$
\begin{aligned}
\vec{y} &= (\alpha\mathbf{P} + \alpha\mathbf{D} + (1-\alpha)\mathbf{E} - \alpha\mathbf{D})\vec{y} + \alpha\vec{v} \\
&= (\alpha\mathbf{P} + (1-\alpha)\mathbf{E})\vec{y} + \alpha\vec{v} \\
&= \alpha\mathbf{P}\vec{y} + (1-\alpha)\mathbf{E}\vec{y} + \alpha\vec{v} \\
&= \alpha\mathbf{P}\vec{y} + \beta\vec{v} + \alpha\vec{v} && \text{where } \beta > 0 \\
&= \alpha\mathbf{P}\vec{y} + \gamma\vec{v} && \text{where } \gamma > 0
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&= \alpha\mathbf{P}\vec{y} + \gamma\vec{v} && \text{where } \gamma > 0
\end{aligned}
$$

- $\vec{z} = \alpha\mathbf{P}\vec{z} + \vec{v}$

A happy ending

Thank you!!